Spin in the Worldline Path Integral in $2+1$ Dimensions

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Abstract

We present a constructive derivation of a worldline path integral for the effective action and the propagator of a Dirac field in $2+1$ dimensions, in terms of spacetime and $SU(2)$ paths. After studying some general properties of this representation, we show that the auxiliary gauge group variable can be integrated, deriving a worldline action depending only on $x(\tau)$, the spacetime paths.
We then show that the functional integral automatically imposes the constraint $\dot{x}^2(\tau) = 1$, while there is a spin action, which agrees with the the one one should expect for a spin-$\frac{1}{2}$ field.

1 Introduction

The spacetime picture for (quantum) spin degrees of freedom and, in particular, of their classical limit, has been a subject of intense research, because of its many interesting subtleties, both from the physical and mathematical points of view.

Phase-space formulations for spin degrees of freedom which, when quantized, lead to the proper dynamics of spinning particles, have been known since some time ago [1]. Those approaches introduce dynamical variables which are group elements, a fact that naturally renders their path-integral

formulation far from trivial [2] (although they avoid the use of Grassmannian variables). The starting point for the known constructions is usually algebraic or geometric. Although, in the end, the usual propagator for a relativistic spinning particle can be recovered, we believe that a more direct approach to the problem is possible: to construct a quantum description for the spin degrees of freedom starting from the knowledge of the propagator for a Dirac field. As a concrete step in that direction, we shall do that here for the case of a Dirac field in $2+1$ dimensions, in the worldline approach [3, 4], where quantum fluctuations are represented as spacetime trajectories in an auxiliary, *proper time*, variable. After some changes of variables within the worldline path integral, we shall obtain a spacetime picture where the geometric interpretation is an emergent property, rather than an *ab initio* ansatz.

We note that the handling of the spin degrees of freedom in this context, is usually achieved by means of the introduction of Grassmann variables [5, 6, 7]. Other methods, which provide a physically more appealing picture of the spinning degrees of freedom have also been considered, like the coherent-state path integrals [8], a historical review of which can be found in [9].

In this paper, we shall show that one can, indeed, construct a worldline path integral at least in $2+1$ dimensions involving only c-number, commuting variables. This preserves the intuitive appeal of the worldline representation, and at the same time has the interesting practical advantage that it is, in principle, much easier to deal with numerically (i.e., on a lattice). We shall derive that worldline representation by following a *c*onstructive approach: as a starting point, we use an old proposal, originally due to Migdal [10] and which has been extensively tested recently [11]. That first-order (phase space) path integral is then transformed into an equivalent one in terms of spacetime paths $x_\mu(\tau)$ and also $g(\tau)$ paths where $g(\tau) \in SU(2)$, introduced by means of an alternative geometric parametrization for the original integral over the canonical momentum variable.

This 'intermediate' worldline representation is studied, in a semiclassical expansion, to shed light on the dynamics of the spin degrees of freedom, both for the free case and for a system in an external gauge field background. We show that the classical equations of motion can be written in terms of the intrinsic geometric properties of the spacetime path (arc length, curvature and torsion). Besides, by calculating the leading contribution to the path integral in that expansion, we show that the modulus of $\dot{x}(\tau)$ should be constant and equal to 1, a result which is later on shown to be exact at the quantum level.

Finally, the transformation properties of the gauge group variables are used in order to evaluate the gauge group path integral, what yields a worldline path integral over spacetime paths only, weighed by an 'effective world-

line action', which selects paths with $\dot{x}^2(\tau) = 1$ and attaching to them a phase given by a spin action.

The organization of this paper is as follows: in section 2, we first review the main properties of the first-order worldline construction. Then, based on that construction, we introduce the gauge group path integral representation, studying the saddle point properties of the path integral over the group variables. In section 3, we use that representation to derive a few results. In particular, we calculate the path integral over the group variables to derive the effective worldline action, and afterwards we evaluate the free propagator.

In Appendix A, we review an alternative representation for the integral over the canonical momentum, which uses Grassmann variables to deal with the matrix structure of the path ordered exponential. It provides an equivalent way of treating the system that does not require the explicit introduction of matrices. We recall it here in order to compare it with the results obtained with the gauge group approach.

2 The method

In this section we shall introduce a novel representation for the effective action and the propagator for a Dirac field in the presence of an external gauge field. Both will involve a sum over two different kinds of paths: $x_\mu(\tau)$, the spacetime paths, and $g(\tau)$, paths of an $SU(2)$ 'gauge group field', related to the evolution of 'internal' degrees of freedom. Afterwards (see section 3), we will show that it is possible to carry out the integration over the auxiliary gauge group variables.

To begin with, we set up our conventions, and briefly review the main features of the first-order formalism, which is used as starting point for further developments. The action S_f for a Dirac field in an Abelian gauge field background A_μ, in $2+1$ Euclidean dimensions has the form:

$$S_f(\bar{\psi}, \psi, A) \;=\; \int d^3x \, \bar{\psi}(\not{D} + m)\psi \;, \tag{1}$$

where

$$\not{D} \equiv \gamma_\mu D_\mu, \; D_\mu = \partial_\mu + ieA_\mu \;, \; \gamma_\mu^\dagger = \gamma_\mu, \; \mu = 1, 2, 3 \;, \tag{2}$$

and e is the coupling constant. The Dirac matrices are chosen to be $\gamma_\mu = \sigma_\mu$, $\mu = 1$, 2, 3, where σ_μ denote the usual Pauli matrices.

Our starting point for the subsequent construction shall be a first-order worldline path integral, whereby the one-loop effective action $\Gamma_f(A)$ (nor-

malized to $\Gamma_f(0) = 0$):

$$\Gamma_f(A) \equiv -\ln\Big[\frac{\det\big(\not{D}+m\big)}{\det\big(\not{\partial}+m\big)}\Big] = -\mathrm{Tr}\ln\big(\not{D}+m\big) + \mathrm{Tr}\ln\big(\not{\partial}+m\big), \tag{3}$$

is represented as [10]:

$$\Gamma_f(A) = \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}x\,\mathcal{D}p\, e^{i\int_0^T d\tau p_\mu(\tau)\dot{x}_\mu(\tau)} \times \mathrm{tr}\Big[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}\Big]\Big[e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]} - 1\Big]. \tag{4}$$

The fermion propagator, $G_f(x,y)$, is given by a similar integral, albeit without the Dirac trace and with different boundary conditions: $x(T) = x$, $x(0) = y$:

$$G_f(x,y) = \int_0^\infty dT\, e^{-mT} \int_{x(0)=y}^{x(T)=x} \mathcal{D}x\,\mathcal{D}p\, e^{i\int_0^T d\tau p_\mu \dot{x}_\mu} \times \Big[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}\Big]\, e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]}. \tag{5}$$

A path integral measure for $\Gamma_f(A)$ that takes into account the boundary conditions, can be formally written as:

$$\Big[\mathcal{D}x\,\mathcal{D}p\Big]_\Gamma \equiv d^3x(0)\Big[\prod_{0<\tau\leq T}\frac{d^3x(\tau)\,d^3p(\tau)}{(2\pi)^3}\Big]\,\delta[x(T)-x(0)], \tag{6}$$

where the δ function, of course, imposes the periodicity condition. Note that the measure is dimensionless (in $\hbar = 1$ units), as it should be. A more symmetric expression may also be written for this measure,

$$\Big[\mathcal{D}x\,\mathcal{D}p\Big]_\Gamma \equiv \Big[\prod_{0\leq\tau\leq T}\frac{d^3x(\tau)\,d^3p(\tau)}{(2\pi)^3}\Big]\,\delta[x(T)-x(0)]\,\delta[p(T)-p(0)], \tag{7}$$

where the extra integration over $p(0)$ is harmless, since it yields 1 because of the new delta function. Note that this means that, if the integration ranges for both phase space variables are equal, then there is periodicity in both of them.

For the propagator, we have instead:

$$\Big[\mathcal{D}x\,\mathcal{D}p\Big]_G \equiv \frac{d^3p(0)}{(2\pi)^3}\prod_{0<\tau<T}\frac{d^3x(\tau)\,d^3p(\tau)}{(2\pi)^3}, \tag{8}$$

which has the dimensions of a (mass)^3, which combined with the (mass)$^{-1}$ of the dT factor, yields the proper mass dimensions to the fermion propagator in coordinate space. In what follows, we shall omit writing explicitly the suffix (Γ or G) to identify the measure, since that shall be clear from the context.

Since the following manipulations will only involve the $p_\mu(\tau)$ integrals, we formally disentangle them from the $x_\mu(\tau)$ integrals by introducing the definition:

$$\begin{aligned} \Gamma_f(A) &= \int_0^\infty \frac{dT}{T}\, e^{-mT} \int_{x(0)=x(T)} \mathcal{D}x \\ &\times \; e^{-S[x(\tau)]} \left\{ e^{-ie \int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]} - 1 \right\} , \end{aligned} \tag{9}$$

and similarly for the propagator:

$$\begin{aligned} G_f(x,y) &= \int_0^\infty dT\, e^{-mT} \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \\ &\times \; D[x(\tau)]\, e^{-ie \int_0^T d\tau \dot{x}_\mu(\tau) A_\mu[x(\tau)]}, \end{aligned} \tag{10}$$

where $S[x(\tau)]$ and $D[x(\tau)]$ are defined by the integrals:

$$e^{-S[x(\tau)]} = \int \mathcal{D}p \, \mathrm{tr}\big[\mathcal{P} e^{-i \int_0^T d\tau \not{p}(\tau)}\big] e^{i \int_0^T d\tau p_\mu(\tau) \dot{x}_\mu(\tau)} \;, \tag{11}$$

and

$$D[x(\tau)] \;=\; \int \mathcal{D}p \left[\mathcal{P} e^{-i \int_0^T d\tau \not{p}(\tau)}\right] \, e^{i \int_0^T d\tau p_\mu(\tau) \dot{x}_\mu(\tau)} \;. \tag{12}$$

Of course, the two previously defined objects are related by:

$$e^{-S[x(\tau)]} = \mathrm{tr}\Big\{ D[x(\tau)] \Big\} \;. \tag{13}$$

From their definitions, these scalar and matrix functionals (resp. (11) and (12)) can be regarded as *functional* Fourier transformations of two particular functionals of $p(\tau)$. Since these functionals are neither Gaussian, nor do they have a simple structure, it is not evident at all how to evaluate their Fourier transformations. In order to tackle that problem, we shall introduce a change of variables in the $p(\tau)$ integral which shall render their integration easier. At the same time, some features of the spin dynamics will become more transparent.

To prepare the ground for that change of variables, we find it convenient to introduce first some notation and conventions. First, we define the (anti-Hermitian) matrices: $\lambda_\mu \equiv \frac{\sigma_\mu}{2i}$, which can be regarded as a basis for $su(2)$, the Lie algebra of $SU(2)$ [12].

Using that basis, for each value of τ we associate to $x_\mu(\tau)$ and $p_\mu(\tau)$ two (anti-Hermitian) elements in $su(2)$: $\chi(\tau)$ and $\pi(\tau)$, respectively, according to the rules:

$$\begin{aligned} \chi(\tau) &= \lambda_\mu \, \chi_\mu(\tau) \, , \quad \chi_\mu(\tau) \;=\; x_\mu(\tau) \\ \pi(\tau) &= \lambda_\mu \, \pi_\mu(\tau) \, , \quad \pi_\mu(\tau) \;=\; 2 \, p_\mu(\tau) \, . \end{aligned} \tag{14}$$

With this notation, $S[x(\tau)]$ adopts a slightly more compact form:

$$\begin{aligned} e^{-S[x(\tau)]} &= \det(\frac{\delta_{\mu\nu}}{2} I) \int \mathcal{D}\pi \; \mathrm{tr}\big[\mathcal{P} e^{\int_0^T d\tau \, \pi(\tau)}\big] \\ &\times \; e^{-i \int_0^T d\tau \mathrm{tr}[\pi(\tau)\dot{\chi}(\tau)]} \, , \end{aligned} \tag{15}$$

where I is the identity operator acting on periodic functions from $[0, T]$. $\det(\frac{\delta_{\mu\nu}}{2} I)$ accounts for the Jacobian under the change $p_\mu \to \pi_\mu$ which is an infinite constant. Of course, the corresponding expression can be constructed for $D[x(\tau)]$, with the only change of omitting the trace.

2.1 Gauge group variables

To proceed, we introduce a parametrization for the integral over $\pi(\tau)$, in terms of gauge group variables. The crucial point here is to note that $\pi(\tau)$ may in fact be regarded as an $SU(2)$ gauge field defined on a 0+1-dimensional 'spacetime' (with τ as the time). We next introduce gauge transformations for this gauge field, which adopt the form:

$$\pi(\tau) \;\to\; \pi^g(\tau) \;\equiv\; g(\tau)\pi(\tau)g^{-1}(\tau) \;+\; g(\tau)\partial_\tau g^{-1}(\tau), \tag{16}$$

where $g(\tau) \in SU(2)$. It should be clear that neither (15) nor the corresponding expression for the propagator are invariant under these transformations, and that they bear no relationship with the gauge symmetry due to coupling to an external gauge field.

Nevertheless, we can still use the Faddeev-Popov (FP) trick of inserting a 1 inside the path integral, corresponding to a gauge fixing for those transformations. Of course, since this is not a gauge-invariant theory, the gauge group integration shall not factorize out of the integral, and the group variable will become dynamical.

The usual expression for the FP '1' becomes, in this case:

$$1 \;=\; \int \mathcal{D}g \, \delta[\mathcal{F}(\pi^g)] \; \Delta_{\mathcal{F}}[\pi], \tag{17}$$

where $\mathcal{F}(\pi) = (\mathcal{F}_\mu(\pi))_{\mu=1}^3$ is the gauge-fixing functional (with three components, its coordinates in the Lie algebra) and $\Delta_\mathcal{F}[\pi]$ its corresponding FP (gauge-invariant) determinant.

Including this 1 into the expression leading to $S[x(\tau)]$, we see that:

$$
\begin{aligned}
e^{-S[x(\tau)]} &= \det(\frac{\delta_{\mu\nu}}{2} I) \int \mathcal{D}g \, \mathcal{D}\pi \; \delta[\mathcal{F}(\pi^g)] \\
&\times \; \Delta_\mathcal{F}[\pi] \, \mathrm{tr}\big[\mathcal{P} e^{\int_0^T d\tau \, \pi(\tau)}\big] \\
&\times \; e^{-i \int_0^T d\tau \mathrm{tr}[\pi(\tau)\dot{\chi}(\tau)]} \; .
\end{aligned}
\tag{18}
$$

Performing now the change of variables: $\pi \to \pi^{g^{-1}}$ in (18) leads to:

$$
\begin{aligned}
e^{-S[x(\tau)]} &= \det(\frac{\delta_{\mu\nu}}{2} I) \int \mathcal{D}g \, \mathcal{D}\pi \; \delta[\mathcal{F}(\pi)] \; \Delta_\mathcal{F}[\pi] \\
&\times \; \mathrm{tr}\big[g^{-1}(T) \mathcal{P} e^{\int_0^T d\tau \, \pi(\tau)} g(0)\big] \\
&\times \; e^{-i \int_0^T d\tau \mathrm{tr}[\pi^{g^{-1}}(\tau)\dot{\chi}(\tau)]} \; .
\end{aligned}
\tag{19}
$$

We have taken into account the properties of invariance under gauge transformations both for the $\pi(\tau)$ measure and the FP determinant, and also the fact that, under the transformation $\pi(\tau) \to \pi^{g^{-1}}(\tau)$, the path-ordered factor behaves as a Wilson line:

$$
\mathcal{P} e^{\int_0^T d\tau \, \pi(\tau)} \;\to\; g^{-1}(T) \mathcal{P} e^{\int_0^T d\tau \, \pi(\tau)} g(0) \; .
\tag{20}
$$

To proceed, we must decide which particular gauge fixing $\mathcal{F}$ to use. Since we only have one spacetime component in $\pi(\tau)$ (note that the $\pi_\mu(\tau)$ are 'internal space' components from the point of view of the τ-spacetime), the simplest possible choice is of course:

$$
\mathcal{F}_\mu \;=\; \pi_\mu \; ,
\tag{21}
$$

a 'temporal gauge', which in this case will erase the gauge field completely, at the expense of introducing $g(\tau)$ as a dynamical variable, since the integrand is not gauge-invariant. Indeed, using (21) into (19), we see that:

$$
\begin{aligned}
e^{-S[x(\tau)]} &= \det(\frac{\delta_{\mu\nu}}{4\pi} I) \, \det(\partial_\tau) \int \mathcal{D}g \, \mathrm{tr}\big[g^{-1}(T) \, g(0)\big] \\
&\times \; e^{-i \int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\dot{\chi}(\tau)]} \; ,
\end{aligned}
\tag{22}
$$

where $\det(\partial_\tau)$ is what remains of the FP determinant when it is evaluated on the gauge-fixed slice $\pi_\mu(\tau) = 0$, and we included the $(2\pi)^{-1}$ factors that came with the definition of the integration measure.

Thus we have obtained the 'intermediate' representation (22) for S, where no path-ordering appears, and the integration variables are $x_\mu(\tau)$ and $g(\tau)$. The two $x_\mu(\tau)$-independent factors may be included into a constant $\mathcal{N}$ so that we shall write:

$$
\begin{aligned}
e^{-S[x(\tau)]} &= \frac{1}{\mathcal{N}} \int \mathcal{D}g \, \mathrm{tr}\big[g^{-1}(T)\, g(0)\big] \\
&\times \;\; e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\dot{\chi}(\tau)]} \; . \qquad (23)
\end{aligned}
$$

2.2 Classical limit

In order to interpret the path integral over $g(\tau)$, leading to the action $S[x(\tau)]$, we shall first consider its saddle point equation, as derived from the leading term in the corresponding semiclassical expansion. Even at this stage, a non trivial condition on the paths shall arise, namely, that the modulus of $\dot{x}(\tau)$ has to be constant at the saddle point. Remarkably, the same condition will emerge at the quantum level, as an exact constraint.

Starting from the defining expression for S:

$$
\begin{aligned}
e^{-S[x(\tau)]} &= \frac{1}{\mathcal{N}} \int \mathcal{D}g \, \mathrm{tr}\big[g^{-1}(T)\, g(0)\big] \\
&\times \;\; \exp\Big\{ -\frac{1}{\hbar}\, I[l,\chi] \Big\} \; , \qquad (24)
\end{aligned}
$$

where we have reinserted an $\hbar$ factor, and we defined the functional:

$$
I[l,\chi] \equiv i \int_0^T d\tau \mathrm{tr}[l(\tau)\dot{\chi}(\tau)] \qquad (25)
$$

with

$$
l(\tau) = \lambda_\mu l_\mu(\tau) = g^{-1}(\tau)\partial_\tau g(\tau) \; . \qquad (26)
$$

To find the saddle points of the above expression with respect to g (χ is here to be regarded as 'external' in the g integral), we consider variations with respect to g. Since such variations should still leave the field in $SU(2)$, we may parametrize them as follows:

$$
g(\tau) \;\equiv\; \tilde{g}(\tau) h(\tau), \qquad (27)
$$

where both $\tilde{g}(\tau)$ and $h(\tau)$ are in $SU(2)$. For $\tilde{g}(\tau)$ we assume it to verify the boundary conditions: $\tilde{g}(0) = g(0)$ and $\tilde{g}(T) = g(T)$, while $h(0) = h(T) = 1$.

To find the extrema of the action, it is sufficient to consider an $h(\tau)$ infinitesimally close to the identity:

$$
h(\tau) \;\simeq\; 1 + \alpha(\tau), \qquad (28)
$$

where $\alpha(\tau)$ is an infinitesimal $su(2)$ matrix verifying

$$\alpha(\tau)^\dagger = -\alpha(\tau), \quad \mathrm{tr}\,\alpha(\tau) \; = \; 0 \; . \tag{29}$$

Under this transformation l behaves as follows:

$$l(\tau) \;\to\; l(\tau) + \delta_\alpha l(\tau), \tag{30}$$

where

$$\delta_\alpha l(\tau) \; = \; D_\tau \alpha(\tau) \equiv \partial_\tau \alpha(\tau) + [l(\tau), \alpha(\tau)] \, . \tag{31}$$

Then the saddle point equation follows from the requirement that

$$\delta_\alpha \int_0^T d\tau \, \mathrm{tr}\big[l(\tau)\dot{\chi}(\tau)\big] = 0, \tag{32}$$

which is equivalent, after an integration by parts, to:

$$0 = -\int_0^T d\tau \, \mathrm{tr}\big[\alpha(\tau) D_\tau \dot{\chi}(\tau)\big] \; . \tag{33}$$

Then the differential equation for the extrema is:

$$D_\tau \dot{\chi}(\tau) = 0 \; , \tag{34}$$

a sort of constant covariance condition which, in components, reads as follows:

$$\ddot{x}_\mu(\tau) \; + \; \epsilon_{\mu\nu\rho} \, l_\nu(\tau) \, \dot{x}_\rho(\tau) \; = \; 0 \; . \tag{35}$$

We insist that the above equation must be thought of as an equation for $\tilde{g}$ (via $\tilde{l}$), and not for $\dot{x}_\mu$, which does indeed appear in the equation, but as an *external* function. However, we do get from the above equation a consistency condition (required in order to have a saddle point). Indeed, contracting with $\dot{x}_\mu$, we get:

$$\dot{x} \cdot \ddot{x} = 0 \Rightarrow \frac{d\dot{x}^2}{d\tau} = 0 \; , \tag{36}$$

i.e. $\dot{x}^2 = v^2 =$ constant. This means that, *in order for a semiclassical limit to exist*, the modulus of $\dot{x}_\mu(\tau)$ has to be a constant, a condition that we shall assume is, indeed, fulfilled.

In the geometric analysis that now follows, we find it convenient to introduce the unit tangent vector $t_\mu \equiv \dot{x}_\mu / v$, ($t^2 = 1$), and to parametrize everything in terms of the arc length s, which is in fact proportional to τ:

$$ds \equiv \sqrt{\dot{x}^2} d\tau = v d\tau \;\Rightarrow\; s \; = \; v\tau \, . \tag{37}$$

Introducing the notation: $f'(s) \equiv \frac{df}{ds}$, we then have:

$$t'_\mu(s) + \epsilon_{\mu\nu\rho}\tilde{l}_\nu(s)t_\rho(s) \;=\; 0 \;, \tag{38}$$

where $\tilde{l}(s) \equiv \tilde{g}^{-1}(s)\partial_s\tilde{g}(s)$. Before proceeding, we note that (38) is exactly like the equation for a 'spin' variable [14], t_μ, in the presence of a time-dependent 'magnetic field' background $\tilde{l}_\mu$. Note, however, that it has to be regarded as an equation determining $\tilde{l}_\mu(s)$. We immediately see that that equation can only fix $\tilde{l}_\mu(s)$ modulo a term proportional to $t_\mu(s)$.
We shall then, for the sake of simplicity, assume that $\tilde{l}_\mu(s)$ is orthogonal to t_μ in (38), keeping in mind that we can afterwards add to the solution an arbitrary term proportional to the tangent vector.

By contracting (38) with $\tilde{l}_\mu$, we also see that it is also orthogonal to t'_μ. In other words, $\tilde{l}$ is orthogonal to n, where $n_\mu(s)$ is the principal normal to the curve $x_\mu(s)$ at the point s.

Since (by assumption) $\tilde{l}_\mu$ is orthogonal to t_μ, we see that $\tilde{l}_\mu$ must, in fact, have the direction of the *binormal* b_μ, which is given by:

$$b_\mu(s) \;=\; \epsilon_{\mu\nu\rho}t_\nu(s)\, n_\rho(s) \;, \tag{39}$$

Thus,

$$\tilde{l}_\mu(s) = \eta(s)b_\mu(s) \tag{40}$$

and the function $\eta(s)$ can be determined from the fact that one of Frenet's equations is just the assertion that $t'(s)$ is equal to the curvature $\kappa(s)$ times $n_\mu(s)$. Since, on the other hand, $\epsilon_{\mu\nu\rho}b_\nu t_\rho = n_\mu$, we see that the equation of motion implies:

$$\tilde{l}_\mu(s) = -\kappa(s)b_\mu(s) \;, \tag{41}$$

and this determines completely the component of $\tilde{l}_\mu$ that is orthogonal to $t_\mu(s)$. This may be put more explicitly in terms of the curve's equation, as follows:

$$\tilde{l}_\mu(s) = -\epsilon_{\mu\nu\rho}x'_\nu(s)\, x''_\rho(s) \;, \tag{42}$$

or, in terms of τ,

$$l_\mu(\tau) \;=\; -v^{-2}\,\epsilon_{\mu\nu\rho}\dot{x}_\nu(\tau)\,\ddot{x}_\rho(\tau) \;. \tag{43}$$

Of course, since it is orthogonal to $\dot{x}_\mu$ it yields no contribution to the value of the functional I at the saddle point. In other words, the saddle-point action is independent the component of $\tilde{l}_\mu$ which is in the direction of the binormal, although that component undergoes a non trivial evolution as a result of that equation.

Then the only contribution comes from the 'longitudinal' part, $\tilde{l}^{\parallel}$ proportional to t_μ:

$$\tilde{l}^{\parallel}_{\mu}(\tau) \;=\; \xi(\tau)\, \dot{x}_\mu(\tau) \;. \tag{44}$$

where ξ parametrizes the arbitrariness related to the fact that a term proportional to the tangent vector may be added to the solution $\tilde{l}$.

Then the saddle point contribution to the action $S[x(\tau)]$ is:

$$e^{-S[x(\tau)]}\big| \;\sim\; \frac{1}{\mathcal{N}} \int \mathcal{D}\xi \, \mathrm{tr}\big[\tilde{g}^{-1}(T)\, \tilde{g}(0)\big] \;\times\; e^{\frac{i}{2}\int_0^T d\tau \xi(\tau) v^2} \;. \tag{45}$$

Besides, $\tilde{g}$ can be of course determined from the knowledge of $\tilde{l}$:

$$\tilde{g}^{-1}(L) \;=\; \mathcal{P} e^{-\int_0^\tau d\tilde{\tau} l(\tilde{\tau})} \tilde{g}^{-1}(0) \tag{46}$$

so that:

$$\mathrm{tr}\big[\tilde{g}^{-1}(T)\, \tilde{g}(0)\big] \;=\; \mathrm{tr}\big[\mathcal{P} e^{\int_0^T d\tau [v^{-2}\epsilon_{\mu\nu\rho}\dot{x}_\nu \ddot{x}_\rho - \xi(\tau)\dot{x}_\mu(\tau)]\lambda_\mu}\big] \;. \tag{47}$$

Thus, the saddle point contribution involves an integration over ξ (it is not determined by the equations) and it adopts the form:

$$e^{-S[x(\tau)]} \;\propto\; \int \mathcal{D}\xi \, \mathrm{tr}\big[\mathcal{P} e^{\int_0^T d\tau [v^{-2}\epsilon_{\mu\nu\rho}\dot{x}_\nu \ddot{x}_\rho - \xi(\tau)\dot{x}_\mu(\tau)]\lambda_\mu}\big] e^{\frac{i}{2}\int_0^T d\tau \xi(\tau) v^2} \;. \tag{48}$$

Let us now perform the exact integration over the 'moduli' ξ; we shall see that this yields a constraint on the constant v, the modulus of $\dot{x}(\tau)$. The functional integral over ξ can be discretized into N ($N \to \infty$) normal integrals, and at each discrete time τ_i we have to evaluate the (normal) integral:

$$\mathcal{I}(\tau_i) \;\equiv\; \int_{-\infty}^{+\infty} \frac{d\xi(\tau_i)}{2\pi} e^{\frac{i}{2}\delta\tau \xi(\tau_i)\left(\dot{x}_\mu(\tau_i)\sigma_\mu + v^2\right)} \;, \tag{49}$$

where $\delta\tau = T/N$. Taking into account the fact that:

$$\begin{aligned} e^{\frac{i}{2}\delta\tau \xi(\tau_i)\dot{x}_\mu(\tau_i)\sigma_\mu} \;&=\; \frac{1+\sigma_\mu t_\mu(\tau_i)}{2}\, e^{\frac{i}{2}\delta\tau\xi(\tau_i)} \\ &+\; \frac{1-\sigma_\mu t_\mu(\tau_i)}{2}\, e^{-\frac{i}{2}\delta\tau\xi(\tau_i)} \;, \end{aligned} \tag{50}$$

we see that the integral over $\xi(\tau_i)$ yields δ functions:

$$\mathcal{I}(\tau_i) \;=\; \frac{2}{\delta\tau}\,\delta(v^2-1)\,\frac{1+\sigma_\mu t_\mu(\tau_i)}{2} + \frac{2}{\delta\tau}\,\delta(v^2+1)\,\frac{1-\sigma_\mu t_\mu(\tau_i)}{2} \;. \tag{51}$$

Since $v > 0$, only the first term survives and coming back to the original expression for the leading semiclassical contribution, we see that:

$$e^{-S[x(\tau)]} \propto \delta(v-1)\mathrm{tr}\Big\{\mathcal{P}\big[e^{\int_0^T d\tau \epsilon_{\mu\nu\rho}\dot{x}_\nu \ddot{x}_\rho \lambda_\mu}(\frac{1+\sigma_\mu t_\mu(\tau)}{2})\big]\Big\}, \tag{52}$$

or

$$e^{-S[x(\tau)]} \propto \delta(v-1)\, e^{-\frac{i}{2}\int_0^T d\tau \kappa(\tau)}\mathrm{tr}\Big\{\mathcal{P}\big[(\frac{1+\sigma_\mu b_\mu(\tau)}{2})(\frac{1+\sigma_\mu t_\mu(\tau)}{2})\big]\Big\}, \tag{53}$$

and

$$e^{-S[x(\tau)]} \propto \delta(v-1)\, e^{-\frac{i}{2}\int_0^T d\tau \kappa(\tau)}\mathrm{tr}\Big\{\mathcal{P}\frac{1+i\sigma_\mu n_\mu(\tau)}{2}\Big\}. \tag{54}$$

Let us finally consider what happens when the external gauge field is included, so that we also have the equations of motion for x, as determined from its electromagnetic coupling:

$$\frac{\delta}{\delta x_\mu(\tau)}\int_0^T d\tau \mathrm{tr}[l(\tau)\dot{\chi}(\tau)] + e\,\dot{x}_\mu(\tau)A_\mu(x(\tau)) = 0, \tag{55}$$

which implies

$$\frac{1}{2}\dot{l}_\mu(\tau) = -\epsilon_{\mu\nu\rho}\, e\tilde{F}_\nu\, \dot{x}_\rho, \tag{56}$$

where we introduced the dual of $F_{\mu\nu}$, $\tilde{F}_\mu = \epsilon_{\mu\nu\rho}F_{\nu\rho}$. We also see that

$$\dot{t}\cdot l = 0, \quad \dot{l}\cdot\tilde{F} = 0, \quad \dot{l}\cdot t = 0. \tag{57}$$

which implies that $t \cdot l =$ constant.

For simplicity let us consider the free case, $e = 0$, Eq. (56) implies $l_\mu =$ constant, and then Eq. (38) gives

$$t_\mu(\tau) = (e^{-\tau L})_{\mu\nu}\, t_\nu(0), \tag{58}$$

where

$$L_{\mu\nu}(\tau) \equiv \epsilon_{\mu\nu\rho}\, l_\rho(\tau). \tag{59}$$

by choosing l to point in the 3 direction $l_\mu(0) = l\delta_{\mu 3}$ we arrive at

$$\dot{t}_3 = 0, \tag{60}$$

and

$$\dot{t}_i = l\,\epsilon_{ij}\, t_j, \quad i,j = 1,2, \tag{61}$$

which gives as a solution

$$t_3 = a = \text{const.}, \quad t_1 = b\cos(|l|\tau + c) \quad t_2 = -b\sin(|l|\tau + c), \tag{62}$$

where a and b are constants, $a^2 + b^2 = 1$, and c is an arbitrary phase. *i.e.* the tangent vector precesses even in the free (no field) case. In coordinate space, this means that the particle describes an helix.

In the general case we have the coupled system of equations (38) and (56) and the additional conditions

$$\dot{x}^2 = \text{constant}, \quad t \cdot l = \text{constant}. \tag{63}$$

3 Consequences

3.1 Calculation of $S[x(\tau)]$

To find $S[x(\tau)]$ is tantamount to obtaining a representation where only $x_\mu(\tau)$ paths remain (with $S[x(\tau)]$ plus the Maxwell interaction term with A as the weight).

We first note that there is an important Ward identity that follows by performing an *infinitesimal* change of variables in the original expression for S. Performing an infinitesimal change of variables $g(\tau) \rightarrow g(\tau)\, h(\tau)$ where $h(\tau) \simeq 1 + \alpha(\tau)$ and $\alpha(0) = \alpha(T) = 0$, we see that:

$$\begin{aligned} e^{-S[x(\tau)]} &= \frac{1}{\mathcal{N}} \int \mathcal{D}g\, \mathrm{tr}\big[g^{-1}(T)\, g(0)\big] \\ &\times \Big(1 + i \int_0^T d\tau \mathrm{tr}[\alpha(\tau) D_\tau \dot{\chi}(\tau)]\Big) e^{-i \int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau) \dot{\chi}(\tau)]} \,, \end{aligned} \tag{64}$$

to first order in α. Since the term independent of α equals the left hand side of the equation, we obtain:

$$\begin{aligned} 0 &= \int \mathcal{D}g\, \mathrm{tr}\big[g^{-1}(T)\, g(0)\big] \\ &\times D_\tau \dot{\chi}(\tau)\; e^{-i \int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau) \dot{\chi}(\tau)]} \,. \end{aligned} \tag{65}$$

Of course, this means that the saddle point equation holds on the average. It may be regarded also as a Ward identity for the path integral, which can be written in components as follows:

$$\begin{aligned} 0 &= \int \mathcal{D}g\, \mathrm{tr}\big[g^{-1}(T)\, g(0)\big] \\ &\times \Big[\ddot{x}_\mu(\tau) \,+\, \epsilon_{\mu\nu\rho}\, l_\nu(\tau)\, \dot{x}_\rho(\tau)\Big]\; e^{-i \int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau) \dot{\chi}(\tau)]} \,. \end{aligned} \tag{66}$$

Note that since $x_\mu(\tau)$ and $\ddot{x}_\mu(\tau)$ are 'external' they can be extracted out of the integral, so that we have .

$$\ddot{x}_\mu(\tau) \,+\, \epsilon_{\mu\nu\rho}\, \langle l_\nu(\tau)\rangle\, \dot{x}_\rho(\tau) \;=\; 0 \,, \tag{67}$$

where the average symbol means:

$$\langle \ldots \rangle \;\equiv\; \frac{\int \mathcal{D}g\,\mathrm{tr}\big[g^{-1}(T)\,g(0)\big]\ldots e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\dot{\chi}(\tau)]}}{\int \mathcal{D}g\,\mathrm{tr}\big[g^{-1}(T)\,g(0)\big]e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\dot{\chi}(\tau)]}} \;. \tag{68}$$

Contracting the previous equations with $x_\mu(\tau)$, we derive the consistency condition for the Ward identity to be valid:

$$\ddot{x}(\tau)\cdot\dot{x}(\tau) = 0 \;, \tag{69}$$

i.e., $|\dot{x}(\tau)|$ must be constant, as when considering the classical limit. The previous equation means that the path integral over g is not even defined when $|\dot{x}(\tau)|$ is not constant, since the Ward identity (related to the invariance of the measure) would then be violated. Hence, the kinematical constraint on $x_\mu(\tau)$ follows in our representation from the underlying symmetry of the gauge-group path integral. We can then conclude that there should be a functional δ factor imposing the previous constraint in S. Namely

$$e^{-S[x(\tau)]} \;=\; \delta[\dot{x}^2(\tau) - v^2]\,e^{-\tilde{S}[x(\tau)]} \;, \tag{70}$$

where v is a constant and $\tilde{S}$ denotes S evaluated for $|\dot{x}| = v$. We shall show at the end of this subsection that in fact $v = 1$, as in the saddle point contribution.

Let as first clean up things a bit by performing a finite change of variables that will decouple a spin contribution

$$g(\tau) \;\equiv\; g(\tau)\,h(\tau) \;, \tag{71}$$

where $h(\tau) \in SU(2)$ is a fixed (g-independent) element whose form shall be determined below. The group measure is invariant:

$$\mathcal{D}(g\,h) \;=\; \mathcal{D}g \;, \tag{72}$$

but all the other ingredients in the path integral do change. Indeed,

$$\begin{aligned} \mathrm{tr}\big[g^{-1}(T)g(0)\big] \;&\to\; \mathrm{tr}\big[g^{-1}(T)g(0)h(0)h^{-1}(T)\big] \;, \\ g^{-1}\partial_\tau g \;&\to\; h^{-1}(g^{-1}\partial_\tau g)h \,+\, h^{-1}\partial_\tau h \;. \end{aligned} \tag{73}$$

Thus we may write:

$$\begin{aligned} e^{-\tilde{S}[x(\tau)]} \;&=\; \frac{1}{\mathcal{N}}\, e^{-i\int_0^T d\tau \mathrm{tr}[h^{-1}(\tau)\partial_\tau h(\tau)\dot{\chi}(\tau)]} \\ &\times\; \int \mathcal{D}g\;\mathrm{tr}\big[g^{-1}(T)g(0)h(0)h^{-1}(T)\big] \\ &\times\; e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\dot{\chi}^{h^{-1}}(\tau)]} \;, \end{aligned} \tag{74}$$

where:

$$\dot{\chi}^{h^{-1}}(\tau) \;=\; h(\tau)\dot{\chi}(\tau)h^{-1}(\tau) \;, \tag{75}$$

and we have extracted out of the integral a factor that is independent of g. The modulus of $\dot{x}$ is (implicitly) assumed to be constant.

Although (74) is valid for any function h, we shall use one that decouples spin from kinetic energy. Since the latter should only depend on the modulus of $\dot{x}_\mu$, the obvious choice is to use an $h(\tau)$ such that $\dot{\chi}^{h^{-1}}(\tau)$ (75) is diagonal. Since $\dot{\chi}$ is anti-Hermitian and traceless, it can be diagonalized by an $SU(2)$ similarity transformation to a matrix proportional to λ_3

$$\dot{\chi}^{h_{\dot{\chi}}^{-1}}(\tau) \;=\; \lambda_3\, v \tag{76}$$

(where the eigenvalues of $\dot{\chi}(\tau)$ are of course $\pm\frac{1}{2i}v$). $h_{\dot{\chi}}$ above is a function of the direction of $\dot{x}(\tau)$, i.e., of the tangent vector t_μ, whose form we shall consider soon.

We know that the $x_\mu(\tau)$ paths we need to consider are periodic in T. Let us also assume that $\dot{x}_\mu(T)$ points in the same direction as $\dot{x}_\mu(0)$, so that the tangent vectors are parallel and pointing in the same direction. Then $h_{\dot{\chi}}(0)\, h_{\dot{\chi}}^{-1}(T) = 1$, and we obtain:

$$\begin{aligned} e^{-\tilde{S}[x(\tau)]} \;&=\; \frac{1}{\mathcal{N}}\; e^{-i\int_0^T d\tau \mathrm{tr}[h_{\dot{\chi}}^{-1}(\tau)\partial_\tau h_{\dot{\chi}}(\tau)\dot{\chi}(\tau)]} \\ &\times\; \int \mathcal{D}g\; \mathrm{tr}\big[g^{-1}(T)g(0)\big] \\ &\times\; e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}(\tau)\partial_\tau g(\tau)\lambda_3 v]} \;, \end{aligned} \tag{77}$$

or, introducing components in the algebra,

$$\begin{aligned} e^{-\tilde{S}[x(\tau)]} \;&=\; \frac{1}{\mathcal{N}}\; e^{\frac{i}{2}\int_0^T d\tau R_\mu(\tau)\dot{x}_\mu(\tau)} \\ &\times\; \int \mathcal{D}g\; \mathrm{tr}\big[g^{-1}(T)g(0)\big] \\ &\times\; e^{\frac{i}{2}\int_0^T d\tau (g^{-1}(\tau)\partial_\tau g(\tau))_3 v} \;, \end{aligned} \tag{78}$$

where:

$$R_\mu(\tau) \;\equiv\; \big(h_{\dot{\chi}}^{-1}(\tau)\partial_\tau h_{\dot{\chi}}(\tau)\big)_\mu \;, \tag{79}$$

is an $\dot{x}_\mu(\tau)$ dependent vector field.

Note that we have managed to decompose the action $\tilde{S}$ into two contributions:

$$\tilde{S}[x(\tau)] \;=\; S_k(v) \;+\; S_h[x(\tau)] \tag{80}$$

where S_k and S_h denote kinetic and spin parts, respectively. Note that the kinetic term is just a function (not a functional). They are defined by:

$$e^{-S_k(v)} = \frac{1}{\mathcal{N}} \int \mathcal{D}g \, \mathrm{tr}\big[g^{-1}(T)g(0)\big] \times e^{\frac{i}{2}\int_0^T d\tau (g^{-1}(\tau)\partial_\tau g(\tau))_3 v} , \tag{81}$$

and

$$S_h[x(\tau)] = -\frac{i}{2}\int_0^T d\tau R_\mu(\tau)\dot{x}_\mu(\tau) . \tag{82}$$

Let us first find a more explicit form for the spin term. To that end, we note that, $\dot{x}_\mu^{h^{-1}}$, the transformed of $\dot{x}_\mu$, differs from it by a local (i.e., time-dependent) rotation. Introducing spherical coordinates for the vector $\dot{x}_\mu$:

$$\dot{x}_\mu = v\big(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta\big) \tag{83}$$

we easily find the form of the matrix $h_{\dot{\chi}}(\tau)$ to be:

$$h_{\dot{\chi}} = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -\sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{pmatrix} . \tag{84}$$

Finally, we obtain the three components of R_μ:

$$R_1 = \dot{\theta}\sin\phi\,, \quad R_2 = -\dot{\theta}\cos\phi\,, \quad R_3 = -\dot{\phi}\,. \tag{85}$$

Inserting this into the expression for the spin term, we find the rather simple result:

$$S_h[x(\tau)] = \frac{i}{2}v\int_0^T d\tau\ \cos\theta\,\dot{\phi}\,, \tag{86}$$

or,

$$S_h[x(\tau)] = \frac{i}{2}v\int d\phi\ \cos\theta\,. \tag{87}$$

It is worth noting that, when $v = 1$ (which will turn out to be the case), (87) is a topological term, whose form agrees exactly with the known proposals for the quantization of a spin-$\frac{1}{2}$ degree of freedom [13]. The integral of the 1-form $d\varphi\cos\theta$ might be converted into the surface integral of a 2-form which is, however, multivalued.

We now show that $v = 1$ and calculate the kinetic term, $S_k[v]$. We already know that $|\dot{x}(\tau)|$ is constant, and by applying a procedure which is also based on performing a change of variables we can show that it has to be quantized, assuming any (positive) integer value.

We now consider in (81) the change of variables:

$$g(\tau) \;\to\; g(\tau)\, h_z(\tau) \;, \tag{88}$$

where $h_z(\tau)$ is an element of $SU(2)$ with the special form:

$$h_z(\tau) \;=\; e^{\frac{i}{2}\,\theta(\tau)\,\sigma_3} \;, \tag{89}$$

such that $\theta(T) - \theta(0) = 4\pi l$, with $l \in \mathbb{Z}$. Due to this boundary condition, we see that:

$$\mathrm{tr}\big[g^{-1}(T)g(0)\big] \;\to\; \mathrm{tr}\big[g^{-1}(T)g(0)\big] \tag{90}$$

and

$$\begin{aligned} & e^{\frac{i}{2}v\int_0^T d\tau (g^{-1}(\tau)\partial_\tau g(\tau))_3} \;\to \\ & e^{\frac{i}{2}v\int_0^T d\tau (g^{-1}(\tau)\partial_\tau g(\tau))_3} \;\times\; e^{-\frac{i}{2}v\int_0^T d\tau\partial_\tau\theta(\tau)} \;. \end{aligned} \tag{91}$$

Since we are just performing a change of variables in one and the same integral, we derive the identity:

$$1 \;=\; e^{-\frac{i}{2}v\int_0^T d\tau\partial_\tau\theta(\tau)} \;. \tag{92}$$

This implies:

$$v \times l \;\in\; \mathbb{Z}\,\forall l \;. \tag{93}$$

Of course, this means that v must be an integer (and positive because it is a modulus). Since, on the other hand, we know that its classical value is $v = 1$, the only possible value is $v = 1$ also for the quantum case. Then

$$e^{-S_k(v)} \;=\; e^{-S_k(1)} \;=\; \text{constant} \;. \tag{94}$$

Putting together the results for kinetic and spin parts, we see that:

$$e^{-S[x(\tau)]} \;=\; \frac{1}{\mathcal{N}'}\,\delta[\dot{x}^2(\tau) - 1]\; e^{-\frac{i}{2}\int d\phi\; \cos\theta} \tag{95}$$

where

$$\frac{1}{\mathcal{N}'} \;\equiv\; \frac{e^{-S_k(1)}}{\mathcal{N}} \;, \tag{96}$$

which is a function of T, and not of the path. Its precise form may be written in terms of the original variables. Indeed, collecting all the factors, we see that:

$$\begin{aligned} \frac{1}{\mathcal{N}'} \;&=\; \int \mathcal{D}p\, e^{i\int_0^T d\tau p_3(\tau)}\mathrm{tr}\big[\mathcal{P}e^{-i\int_0^T d\tau \not{p}(\tau)}\big] \\ &=\; e^{-\tilde{S}[x(\tau)]}\big|_{x_\mu(\tau)\,=\,\delta_{\mu 3}}\; F(T) \;, \end{aligned} \tag{97}$$

namely, it is determined by the action corresponding for path that corresponds to a straight line with speed 1 pointing in the third direction.

We then arrive at our final expression

$$\begin{aligned}\Gamma_f(A) &= \int_0^\infty \frac{dT}{T}\, e^{-mT}\, F(T) \int_{x(0)=x(T)} \mathcal{D}x\; \delta[\dot{x}^2(\tau)-1] \\ &\times\; e^{-\frac{i}{2}\int d\phi\;\cos\theta}\; e^{-ie\int_0^T d\tau \dot{x}_\mu(\tau)A_\mu[x(\tau)]}\;. \end{aligned} \tag{98}$$

Regarding the fermion propagator, the same constraint on the modulus of $\dot{x}$ is obtained, and the resulting expression for $D[x(\tau)]$ is:

$$D[x(\tau)] \;=\; \frac{1}{\mathcal{N}}\delta[\dot{x}^2(\tau)-1]\; h_{\dot{\chi}}^{-1}(T)\; h_{\dot{\chi}}(0)\; e^{-\frac{i}{2}\int_0^T d\tau\dot{\phi}\cos\theta}\;, \tag{99}$$

where the matrix factors $h_{\dot{\chi}}^{-1}(T)\; h_{\dot{\chi}}(0)$ cannot be taken out of the integral over $x(\tau)$, since they depend not only on the (fixed) boundary values for $x(\tau)$, but also on the derivatives at the boundaries (which are not fixed).

In conclusion, we have succeeded in constructing a spacetime worldline path integral with the spin degrees of freedom, which have been exactly integrated in 2+1 dimensions. The resulting geometric picture had been qualitatively anticipated, in the context of a superspace formulation, see e.g. [15].

The propagator in a constant external field and the issue of the parity violation have been computed in this first order formalism with the Migdal's factorization [11]. Our main result there was that the trace anomaly and the related Chern–Simons current did not require any extra regularization, and are free of any ambiguity, in contrast to other approaches. The lengthy computation should therefore carry through in the special parameterization without any problem, as we illustrate below with the free propagator.

3.2 The free propagator

As a simple application to test the representation, let us perform a derivation, of the free ($e=0$) fermion propagator (the free action is of course an uninteresting constant).

$$\begin{aligned} G(x,y)|_{e=0} &= \det(\frac{\delta_{\mu\nu}}{2})\,\det(\partial_\tau)\int \mathcal{D}x\,\mathcal{D}g\,\left[g^{-1}(T)\,g(0)\right] \\ &\times\; e^{-i\int_0^T d\tau \mathrm{tr}[g^{-1}\partial_\tau g(\tau)\dot{x}(\tau)]}\;. \end{aligned} \tag{100}$$

Of course, we can integrate out $x_\mu(\tau)$, what yields a δ-functional

$$\int \mathcal{D}x\, e^{-i\int_0^T d\tau \mathrm{tr}(g^{-1}\partial_\tau g\dot{x})} \;=\; \prod_{\mu=1}^{3}\left\{\delta[\frac{1}{2}\dot{x}_\mu(\tau)]\right\}$$

$$= \det(2\delta_{\mu\nu}) \prod_{\mu=1}^{3} \left\{ \delta[\partial_\tau (g^{-1}\partial_\tau g)_\mu] \right\} , \tag{101}$$

where the factor 2 comes from the fact that:

$$\mathrm{tr}(g^{-1}\partial_\tau g \dot{x}) = -\frac{1}{2} \, (g^{-1}\partial_\tau g)_\mu \, \dot{x}_\mu \; . \tag{102}$$

Then

$$\begin{aligned} G(x,y)|_{e=0} \; &= \; \det(\partial_\tau) \, \int_0^\infty dT \, e^{-mT} \, \mathcal{D}g \, \delta\big[\partial_\tau (g^{-1}\partial_\tau g)\big] \\ &\quad \big[g^{-1}(T) \, g(0)\big] \, e^{-ir_\mu (y-x)_\mu} \; , \end{aligned} \tag{103}$$

where we have defined:

$$r \; \equiv \; g^{-1}\partial_\tau g \tag{104}$$

an (antihermitian) element of $su(2)$ which can take an arbitrary constant value, by virtue of the δ-functional that appears due to the integration over $x_\mu(\tau)$. On the other hand, since (104) may be solved for $g(\tau)$:

$$g(\tau) \; = \; g(0) \, e^{\tau \, r} \; , \tag{105}$$

we can write:

$$\begin{aligned} &\prod_{\mu=1}^{3} \left\{ \delta[\partial_\tau (g^{-1}\partial_\tau g)_\mu] \right\} \\ &= \; \overset{-1}{\det}(\partial_\tau) \, \int \frac{d^3 r}{(2\pi)^3} \, \delta[g(\tau) - g(0) \, e^{\tau \, r}] \; . \end{aligned} \tag{106}$$

Using this expression we finally obtain:

$$G(x,y)|_{e=0} \; = \; \int \frac{d^3 r}{(2\pi)^3} \frac{e^{ir\cdot(x-y)}}{\not{r} + m} \; . \tag{107}$$

4 Conclusions

We have thus completed a rigorous path integral representation in the world line of effective actions and propagators of spinors in an external gauge field, in the conventional first time formulation and solely in terms of ordinary (commuting) variables. The constructive proof uses gauge theory technicalities which are also understood in terms of geometric and physical principles. This offers a simple description of the classical limit of the quantum spin physics from the Dirac equation, as Feynman started looking for [19]. Beyond the conceptual interest, our formulation have practical applications

for discretizations in general and specially in the world line. In this formalism progress has been achieved in non-perturbative computations, like quenched all order results [18], pair creation and Casimir energies, but restricted to scalar cases or very simple(one dimensional) external fields [20, 21]. Of course, the method has to be extended to four dimensions [16] which is more complicated as the functional integral corresponds to a hypersurface and involves groups with more Cartan elements. Our methods should be complementary to the second order attempts, where instead of the modulus 1 velocity (which can be related to reparametrization invariance), the world lines velocities are Gaussian distributed with special properties [22].

Appendix A: Grassmann variables

Grassmann variables can be introduced to get an alternative expression to the integral over gauge group variables. Indeed, one may recall, for example, the approach introduced by Kleinert [17], to represent the evolution operator for a spin-$\frac{1}{2}$ variable in a time dependent magnetic field background $\mathbf{B}(\tau)$. By introducing three real Grassmann variables θ_μ, and performing the identifications: $B_\mu(\tau) \equiv -2\, p_\mu(\tau)$, $(\mu = 1, 2, 3)$ we see that:

$$e^{-S[x(\tau)]} \;=\; \int_{\theta(T)=-\theta(0)} \mathcal{D}p\, \mathcal{D}\theta\, e^{-\int_0^T d\tau \left[\frac{1}{4}\theta_\mu\dot{\theta}_\mu - \frac{1}{2}\epsilon_{\nu\nu\lambda}p_\mu\theta_\nu\theta_\lambda - i\, p_\mu\dot{x}_\mu\right]} \,, \tag{108}$$

where the antiperiodicity for the Grassmann variables is necessary in order to get the trace of the path-ordered exponential. The other object, $D[x(\tau)]$ is instead given by

$$D[x(\tau)] \;=\; \int_{\theta_\mu(0)=\theta_\mu^{(i)}}^{\theta_\mu(T)=\theta_\mu^{(f)}} \mathcal{D}p\, \mathcal{D}\theta\, e^{-\int_0^T d\tau \left[\frac{1}{4}\theta_\mu\dot{\theta}_\mu - \frac{1}{2}\epsilon_{\nu\nu\lambda}p_\mu\theta_\nu\theta_\lambda - i\, p_\mu\dot{x}_\mu\right]} \,, \tag{109}$$

where the boundary values for θ ($\theta^{(i)}$ and $\theta^{(f)}$) can be chosen in order to evaluate the desired matrix element of the fermion propagator, by selecting the appropriate coherent states at the initial and final times.

By proceeding with the analogy with the system corresponding to a spin in a magnetic field [17], an interesting conclusion may be drawn from the (quantum) equations of motion for the corresponding operators. The relevant dynamical operators are the 'spin' $\hat{S}_\mu$

$$\hat{S}_\mu \;\equiv\; -\frac{i}{4}\varepsilon_{\mu\nu\lambda}\theta_\nu\theta_\lambda, \tag{110}$$

and the velocity operator $\hat{\dot{x}}_\mu$. The equations of motion for the spin degrees of freedom are:

$$\dot{\hat{S}}_\mu(\tau) \;=\; -\epsilon_{\mu\nu\lambda}\,\pi_\nu(\tau)\,\hat{S}_\lambda(\tau), \tag{111}$$

and the constraint equation for p_μ, which is a Lagrange multiplier:

$$\hat{\dot{x}}_\mu(\tau) \;=\; 2\hat{S}_\mu(\tau)\;. \tag{112}$$

Then we see that:

$$\dot{\hat{\dot{x}}}_\mu(\tau) \;=\; -\epsilon_{\mu\nu\lambda}\,\pi_\nu(\tau)\,\hat{\dot{x}}_\lambda(\tau)\;, \tag{113}$$

consistent with the semiclassical equations.

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References

[1] See, for example: A. P. Balachandran, G. Marmo, B. Skagerstam and A. Stern, *Gauge Symmetries and Fibre Bundles*, Springer, Berlin (1983), and references therein.

[2] P. Orland, Int. J. Mod. Phys. A4, 3615 (1989).

[3] R. P. Feynman, Phys. Rev. **76**, 749 (1949).

[4] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[5] Z. Bern and D.A. Kosower, Nucl. Phys. **B362**, 389 (1991); **B379**, 451 (1992).

[6] M.J. Strassler, Nucl. Phys. **B385**, 145 (1992).

[7] C. Alexandrou, R. Rosenfelder and A. W. Schreiber, Phys. Rev. A **59**, 1762 (1999)

[8] M. Stone, Phys.Rev.Lett. 63 (1989) 731.

[9] A. Frydryszak, hep-th/9601020.

[10] A. A. Migdal, Nucl. Phys. **B265** [FSI5], 594 (1986). See also, M.B. Halpern, A. Jevicki, and P. Senjanovic, Phys. Rev. **D16** (1977) 2476; M.B.Halpern and W. Siegel Phys. Rev. D **16** (1977) 2486.

[11] C. D. Fosco, J. Sánchez-Guillén, and R. A. Vázquez, Phys. Rev. D **69** 105022 (2004); arXiv:hep-th/0511026.

[12] We follow the conventions of: V. Rubakov, *Classical Theory of Gauge Fields*, Princeton University Press (2002).

[13] P. Wiegmann, Phys. Rev. Lett. 60, 821 (1988),
E. Fradkin and M. Stone, Phys. Rev. B **38** 10 (1988).

[14] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford Science Publications, 4th. Ed., (2002).

[15] A. M. Polyakov, *Gauge field and strings*, Harwood Academic Publishers, London 1987, Eq. (9.323) and footnote therein.

[16] C. D. Fosco, J. Sánchez-Guillén, and R. A. Vázquez, work in progress.

[17] H. Kleinert, Path integrals in quantum mechanics, statistics, and polymer physics, World Scientific, Singapore, 1995.

[18] H. Gies, J. Sanchez-Guillen, and R.A. Vazquez, JHEP 0508, 067(2005)

[19] R. P. Feynman, Phys. Rev. 84 (1951) 108. See App. D

[20] G. V. Dunne and C. Schuber, Phys.Rev. D72(2005)105004

[21] G. V. Dunne, H. Gies, C. Schubert, and Q. Wang, Phys.Rev. D73 (2006) 065028.

[22] H. Gies and J. Hammerling, Phys.Rev. D72 (2005) 065018.